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Completely monotone functions and the Wallis ratio

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ABSTRACT

The aim of the paper is to improve known estimates of the Wallis ratio. Moreover, we show that these improvements are valid, because certain functions involving the continuous version of the Wallis ratio are completely monotone.

1. Introduction

The Wallis ratio

$$P_n = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}$$

has important applications in pure mathematics (combinatorics, number theory, probabilities) or in other branches of science such as applied statistics, statistical physics and quantum mechanics. It is closely related to the Euler gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0,$$

since

$$P_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}. \tag{1.1}$$

Kazarinoff [1] first proved

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \tag{1.2}$$

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and later on, other accurate estimates were stated. We refer for example to the work by Zhao De Jun [2] with a proof of the following inequalities:

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2}\right)}} < P_n < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/3}\right)}}. \tag{1.3}$$

Zhao and Wu [3] improved the upper bound of the previous inequality, showing that for $0<\varepsilon<1/2$, it holds

$$P_n < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2+\varepsilon}\right)}}, \tag{1.4}$$

whenever $n \ge n^*(\varepsilon)$, where $n^*(\varepsilon)$ is the maximal root of the equation

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

In fact,

$$n^*(\varepsilon) = \frac{\sqrt{(1-2\varepsilon)\left(124\varepsilon^2 - 382\varepsilon - 8\varepsilon^3 + 289\right)} - \left(32\varepsilon + 4\varepsilon^2 - 17\right)}{64\varepsilon}.$$

The first aim of this paper is to prove that the Zhao–Wu inequality (1.4) holds for every $n \ge n^{\#}(\varepsilon)$, where

$$n^{\#}(\varepsilon) = \frac{1}{32\varepsilon}\left(-16\varepsilon - 4\varepsilon^2 + 9\right).$$

Our bound $n^{\#}(\varepsilon)$ is much better than $n^*(\varepsilon)$, having in mind that $n^*(\varepsilon) > n^{\#}(\varepsilon)$, for every $0<\varepsilon<1/2$ and moreover, $n^*(\varepsilon) - n^{\#}(\varepsilon)$ tends to infinity, as $\varepsilon \to 0_+$. However, as we can see in the next section, this bound can sometimes be further improved.

Inequalities (1.3)–(1.4) show us that the best approximations of the form

$$P_n \approx \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-a}\right)}}, \quad a \in \mathbb{R} \tag{1.5}$$

are obtained for $a = 1/2$. Moreover, if we are interested to obtain further accurate approximations of the form

$$P_n \approx \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\theta_n}\right)}}, \quad \theta_n \in \mathbb{R},$$

then θ_n should tend to $1/2$, as n approaches infinity. Such approximations are motivated by

$$\frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-\frac{1}{2}+\frac{3}{16n+\frac{15}{4n}}}\right)}} < P_n < \frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-\frac{1}{2}+\frac{3}{16n}}\right)}}, \tag{1.6}$$

which improves the inequalities in (1.2)–(1.4).

Thanks to (1.1), relation (1.5) can be equivalently written as

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \approx \frac{1}{\sqrt{n\left(1+\frac{1}{4n-a}\right)}}$$

and we are entitled to study the logarithmically completely monotonicity of the functions

$$g_a(x) = \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\sqrt{x\left(1+\frac{1}{4x-a}\right)}, \quad a \in \mathbb{R}.$$

More precisely, we prove that for all $1/2 \le a \le 2$, the function g_a is logarithmically completely monotonic on $(0,\infty)$. Afterwards we exploit $a = 1/2$ case to establish the following double inequality

$$\frac{\alpha}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} \le \frac{\beta}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}}, \tag{1.7}$$

where the constants $\alpha = 1$ and $\beta = \frac{3\sqrt{7\pi}}{14} = 1.0049\ldots$ are the best possible.

2. On the range of the Zhao–Wu inequality

Inequality (1.4) can be studied by defining the sequence

$$x_n = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \varepsilon}\right)}.$$

Indeed, x_n converges to 1, and if it is strictly increasing, then evidently, $x_n < 1$, and (1.4) follows. In this sense, notice that

$$\begin{aligned} \left(\frac{x_{n+1}}{x_n}\right)^2 - 1 &= \frac{8\varepsilon \left(n - n^{\#}(\varepsilon)\right)}{n(n+1)(8n+2\varepsilon+1)(8n+2\varepsilon+7)} \\ &> 0, \end{aligned}$$

as soon as $n > n^{\#}(\varepsilon)$. Thus, we immediately obtain the following properties of the sequence x_n.

Theorem 1. *The sequence*

$$x_n = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \varepsilon}\right)}, \quad n > n^{\#}(\varepsilon)$$

is strictly increasing and for all $n > n^{\#}(\varepsilon)$, it holds

$$\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} < \frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \varepsilon}\right)}}.$$

Using this fact, we obtain the following estimate of the Wallis ratio.

Theorem 2. *For all integers $n \geq 1$, it holds*

$$\frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n + \frac{15}{4n}}}\right)}} < P_n < \frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n}}\right)}}. \tag{2.1}$$

Proof. The sequences

$$u_n = P_n \sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n + \frac{15}{4n}}}\right)}, \qquad v_n = P_n \sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n}}\right)}$$

converge to 1, so it suffices to show that u_n is strictly decreasing and v_n is strictly increasing. In this sense, we have

$$\begin{aligned} \left(\frac{u_{n+1}}{u_n}\right)^2 - 1 &= -\frac{11\,025}{4\left(144n + 64n^2 + 512n^3 + 15\right)} \\ &< 0 \end{aligned}$$

and

$$\begin{aligned} \left(\frac{v_{n+1}}{v_n}\right)^2 - 1 &= \frac{225}{4n(n+1)\left(120n + 64n^2 + 59\right)\left(8n + 64n^2 + 3\right)} \\ &> 0, \end{aligned}$$

so $u_{n+1} < u_n$ and $v_{n+1} > v_n$ and the conclusion follows. $\square$

Now we can see that our bound $n^{\#}(\varepsilon)$ can be improved. By Theorem 2, inequality (1.4) holds as soon as

$$\frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \frac{3}{16n}}\right)}} < \frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n - \frac{1}{2} + \varepsilon}\right)}},$$

that is $n > n^{\&}(\varepsilon) := \frac{3}{16\varepsilon}$. As

$$n^{\&}(\varepsilon) - n^{\#}(\varepsilon) = \frac{1}{32\varepsilon}\left(16\varepsilon + 4\varepsilon^2 - 3\right),$$

it results that the bound $n^{\&}(\varepsilon)$ is better than $n^{\#}(\varepsilon)$, whenever $0 < \varepsilon < \frac{1}{2}\sqrt{19} - 2 = 0.17\,945\ldots$.

Recalling formula (1.1), we can state the following continuous version of estimate (2.1).

Theorem 3. *For all real numbers $x \geq 1$, it holds*

$$\frac{1}{\sqrt{x\left(1+\frac{1}{4x-\frac{1}{2}+\frac{3}{16x+\frac{15}{4x}}}\right)}} < \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} < \frac{1}{\sqrt{x\left(1+\frac{1}{4x-\frac{1}{2}+\frac{3}{16x}}\right)}}.$$

Proof. We use the following inequalities, for all $x \geq 1$:

$$a(x) < \ln\Gamma(x) < b(x),$$

where

$$a(x) = \ln\sqrt{2\pi} + \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7},$$

$$b(x) = \ln\sqrt{2\pi} + \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}.$$

See, e.g. [4, Proof of Theorem 2.1]. Under these hypotheses, we have

$$a\left(x+\frac{1}{2}\right) - b(x+1) < \ln\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} < b\left(x+\frac{1}{2}\right) - a(x+1)$$

and it suffices to show that $y < 0$ and $z > 0$, where

$$y(x) = b\left(x+\frac{1}{2}\right) - a(x+1) + \frac{1}{2}\ln x + \frac{1}{2}\ln\left(1+\frac{1}{4x-\frac{1}{2}+\frac{3}{16x}}\right),$$

$$z(x) = a\left(x+\frac{1}{2}\right) - b(x+1) + \frac{1}{2}\ln x + \frac{1}{2}\ln\left(1+\frac{1}{4x-\frac{1}{2}+\frac{3}{16x+\frac{15}{4x}}}\right).$$

We have

$$y''(x) = -\frac{P(x)}{210x^2(2x+1)^7(x+1)^9\left(64x^2-8x+3\right)^2\left(8x+64x^2+3\right)^2},$$

$$z''(x) = \frac{Q(x)}{210x^2(x+1)^7(2x+1)^9\left(144x+64x^2+512x^3+15\right)^2\left(144x-64x^2+512x^3-15\right)^2},$$

where

$$\begin{aligned}P(x) = {} & 8505 + 195\,615x + 2600\,235x^2 + 31\,504\,293x^3 + 331\,983\,609x^4 \\ & + 2823\,467\,223x^5 + 19\,022\,691\,107x^6 + 103\,154\,678\,365x^7 \\ & + 449\,761\,791\,249x^8 + 1558\,158\,705\,248x^9 + 4225\,609\,480\,226x^{10} \\ & + 8846\,248\,294\,916x^{11} + 14\,124\,164\,423\,840x^{12} + 16\,987\,573\,248\,800x^{13} \\ & + 15\,145\,280\,913\,952x^{14} + 9774\,311\,235\,136x^{15} + 4404\,322\,291\,712x^{16} \\ & + 1309\,721\,534\,464x^{17} + 232\,243\,200\,000x^{18} + 18\,579\,456\,000x^{19}\end{aligned}$$

and

$$\begin{aligned}Q(x) = {} & 680\,731\,072\,234\,078\,005 + 10\,230\,884\,234\,529\,108\,780\,(x-1) \\ & + 73\,172\,105\,412\,805\,709\,390\,(x-1)^2 + 331\,111\,880\,852\,696\,162\,762\,(x-1)^3 \\ & + 1063\,110\,681\,485\,992\,076\,762\,(x-1)^4 + 2574\,237\,428\,828\,093\,653\,415\,(x-1)^5 \\ & + 4877\,363\,074\,768\,772\,414\,477\,(x-1)^6 + 7403\,948\,426\,422\,283\,243\,648\,(x-1)^7\end{aligned}$$

$$+ 9145\,336\,638\,007\,293\,087\,638\,(x-1)^8 + 9282\,184\,400\,266\,314\,999\,188\,(x-1)^9$$
$$+ 7783\,858\,696\,868\,171\,454\,840\,(x-1)^{10} + 5402\,971\,302\,151\,716\,075\,216\,(x-1)^{11}$$
$$+ 3099\,431\,636\,812\,751\,571\,744\,(x-1)^{12} + 1461\,909\,478\,560\,289\,760\,960\,(x-1)^{13}$$
$$+ 561\,750\,658\,911\,633\,367\,424\,(x-1)^{14} + 173\,330\,495\,134\,383\,838\,976\,(x-1)^{15}$$
$$+ 42\,020\,041\,650\,945\,228\,800\,(x-1)^{16} + 7743\,341\,503\,411\,060\,736\,(x-1)^{17}$$
$$+ 1029\,224\,093\,656\,285\,184\,(x-1)^{18} + 90\,098\,141\,173\,383\,168\,(x-1)^{19}$$
$$+ 4311\,467\,620\,827\,136\,(x-1)^{20} + 64\,456\,990\,130\,176\,(x-1)^{21}.$$

Now y is strictly concave and z is strictly convex on $[1,\infty)$, with $y(\infty) = z(\infty) = 0$, so $y(x) < 0$ and $z(x) > 0$, for all $x \in [1,\infty)$. This completes the proof. □

3. Completely monotone functions

The logarithmic derivative of the gamma function ψ is called the digamma (or psi) function, while the derivatives $\psi', \psi'', \psi''', \ldots$ are known as polygamma functions. In what follows, we use the following integral representations, for every positive integer n,

$$\psi^{(n)}(x) = (-1)^{n-1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}} dt \tag{3.1}$$

and for every $r > 0$,

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt. \tag{3.2}$$

See, e.g., [5].

Recall that a function w is completely monotonic in an interval I if w has derivatives of all orders in I such that

$$(-1)^n w^{(n)}(x) \geq 0, \tag{3.3}$$

for all $x \in I$ and $n = 0, 1, 2, 3 \ldots$. Dubourdieu [6] proved that if a non constant function w is completely monotonic, then strict inequalities hold in (3.3). Completely monotonic functions involving $\ln \Gamma(x)$ are important because they produce sharp bounds for the polygamma functions. The famous Hausdorff–Bernstein–Widder theorem [7, p. 161] states that w is completely monotonic on $[0,\infty)$ if and only if

$$w(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a non-negative measure on $[0,\infty)$ such that the integral converges for all $x > 0$.

By a logarithmically completely monotonic function t, we mean a positive function t such that $\ln t$ is completely monotonic.

Related to approximation (1.5), we prove the following

Theorem 4. *For every $1/2 \leq a \leq 2$, the function*

$$g_a(x) = \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \sqrt{x\left(1+\frac{1}{4x-a}\right)}$$

is logarithmically completely monotonic on $[1,\infty)$.

First, we state an auxiliary lemma.

Lemma 1. *Let us define*

$$x_n(a) = (a+5)^n + a^n + 2\cdot 3^n - (a+1)^n - (a+4)^n - 5^n - 1.$$

Then $x_n(a) \geq 0$, for all $n \geq 3$ and $a \in [1/2, 2]$.

Proof. First notice that

$$x_3(a) = 24\left(a-\frac{1}{2}\right), \qquad x_4(a) = 48\left(a-\frac{1}{2}\right)^2 + 288\left(a-\frac{1}{2}\right)^2 + 36,$$

so it suffices to consider only the $n \geq 5$ case. Evidently

$$a^n + 2\cdot 3^n > (a+1)^n,$$

since $3^n \geq (a+1)^n$, then we prove

$$(a+5)^n > (a+4)^n + 5^n + 1, \quad n \geq 5.$$

This inequality can be justified as follows

$$\begin{aligned}\left(\frac{a+4}{a+5}\right)^n + \left(\frac{5}{a+5}\right)^n + \left(\frac{1}{a+5}\right)^n &\leq \left(\frac{a+4}{a+5}\right)^5 + \left(\frac{5}{a+5}\right)^5 + \left(\frac{1}{a+5}\right)^5 \\ &= 1 - \frac{5R(a)}{16\,(a+5)^5} \\ &< 1,\end{aligned}$$

where

$$R(a) = 197 + 8080\left(a - \frac{1}{2}\right) + 2408\left(a - \frac{1}{2}\right)^2 + 320\left(a - \frac{1}{2}\right)^3 + 16\left(a - \frac{1}{2}\right)^4. \quad \square$$

Proof of Theorem 4. We have to prove that $f_a = \ln g_a$ is completely monotonic. In this sense, we have

$$f_a(x) = \ln\Gamma\left(x + \frac{1}{2}\right) - \ln\Gamma\,(x+1) + \frac{1}{2}\ln x + \frac{1}{2}\ln\left(1 + \frac{1}{4x-a}\right),$$

with

$$f_a''(x) = \psi'\left(x + \frac{1}{2}\right) - \psi'\,(x+1) - \frac{1}{2x^2} + \frac{1}{2\left(x - \frac{a}{4}\right)^2} - \frac{1}{2\left(x + \frac{1-a}{4}\right)^2}.$$

Using (3.1)–(3.2), we get

$$\begin{aligned}f_a''(x) &= \int_0^\infty \frac{te^{-t\left(x+\frac{1}{2}\right)}}{1-e^{-t}}dt - \int_0^\infty \frac{te^{-t(x+1)}}{1-e^{-t}}dt - \frac{1}{2}\int_0^\infty te^{-tx}dt + \frac{1}{2}\int_0^\infty te^{-t\left(x-\frac{a}{4}\right)}dt - \frac{1}{2}\int_0^\infty te^{-t\left(x+\frac{1-a}{4}\right)}dt \\ &= \frac{1}{2}\int_0^\infty \varphi\left(\frac{t}{4}\right)\frac{te^{-tx}}{e^t-1}dt,\end{aligned}$$

where $\varphi(t) = 2e^{2t} - e^{at} - e^{4t} + e^{(a-1)t} - e^{(a+3)t} + e^{(a+4)t} - 1$. But

$$\varphi(t) = e^{-t}\sum_{n=3}^{\infty} x_n(a)\frac{t^n}{n!} > 0,$$

and by the above mentioned Hausdorff–Bernstein–Widder theorem, f_a'' is completely monotonic.

Now f_a' is strictly increasing (since $f_a'' > 0$) with $f_a'(\infty) = 0$, so $f_a' < 0$.

Finally, f_a is strictly decreasing (since $f_a' < 0$) with $f_a(\infty) = 0$, so $f_a > 0$ and consequently f_a is completely monotonic. $\square$

In particular, $f_{1/2}$ is strictly decreasing. For all $x \geq 1$, we have

$$0 = f_{1/2}(\infty) < f_{1/2}(x) \leq f_{1/2}(1) = \ln\frac{3\sqrt{7\pi}}{14},$$

which is (1.7).

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